

Non-Abelian Sine-Gordon Solitons

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Abstract

We point out that non-Abelian sine-Gordon solitons stably exist in the $U(N)$ chiral Lagrangian. They also exist in a $U(N)$ gauge theory with two N by N complex scalar fields coupled to each other. One non-Abelian sine-Gordon soliton can terminate on one non-Abelian global vortex. They are relevant in chiral Lagrangian of QCD or in color-flavor locked phase of high density QCD, where the anomaly is suppressed at asymptotically high temperature or density, respectively.

I. INTRODUCTION

Sine-Gordon kinks (solitons) [1] appear in broad range of physics from classical and quantum field theories [2, 3], QCD [4], conformal field theories, integrable systems, and cosmology [5] to condensed matter physics. Condensed matter systems offer a lot of examples of sine-Gordon kinks which can be observed in laboratory experiments, such as Josephson junctions of two superconductors [6], those in multi-layer high T_c superconductors [7], two-gap superconductors [8–10], chiral p-wave superconductors [11], coherently coupled two-component Bose-Einstein condensates (BECs) [12], two separated BECs with a Josephson coupling [13], helium 3 superfluids [14], and ferromagnets [15]. In particular, sine-Gordon kinks are Josephson vortices in Josephson junctions appearing when a magnetic field is applied parallel to a Josephson junction or layers of high T_c superconductors [6, 7, 13, 16]. Another interesting case is that a sine-Gordon kink connects two fractional vortices winding around different components, to constitute a vortex molecule in multi-gap superconductors [10, 17, 18] and coherently coupled multi-component BECs [19–21].

Sine-Gordon kinks also explain relations between topological defects or solitons in different dimensions. Sine-Gordon kinks inside the world-volume of a topological defect represent some other topological defects in the bulk; Sine-Gordon kinks inside a domain wall are vortices, lumps or baby Skyrmions in the bulk [16, 22–24], which explains a relation between sine-Gordon kinks and $\mathbb{C}P^1$ instantons [25, 26]. Sine-Gordon kinks inside a domain wall ring are baby Skyrmions [23]. They represent Skyrmions in the bulk if residing in a domain wall within a domain wall [27–29] or in a vortex string [30, 31], they are Hopfions in the bulk if residing in a toroidal domain wall [32], and are Yang-Mills instantons in the bulk if residing inside a monopole string in Yang-Mills theory in $d = 4 + 1$ dimensions [33].

There have been many proposal of generalizations of the sine-Gordon model. One of such is a complex sine-Gordon model describing a vortex motion in superfluids [34], the $O(4)$ model [35], conformal field theories [36], and a domain wall junction [38]. There have been non-Abelian generalizations such as the matrix sine-Gordon model [39], the symmetric space sine-Gordon model [40] and so on.

In this paper, we discuss yet another non-Abelian generalization of sine-Gordon kinks. We point out that a non-Abelian sine-Gordon kink admitted in the $U(N)$ chiral Lagrangian [37] is a non-Abelian soliton carrying non-Abelian moduli $\mathbb{C}P^{N-1} \simeq SU(N)/[SU(N-1) \times U(1)]$.

Here, the term “non-Abelian” is used in the same way with that of non-Abelian vortices [41–44] carrying non-Abelian $\mathbb{C}P^{N-1}$ moduli, see Refs. [45–47] for a review. As in the same manner with a non-Abelian vortex with non-Abelian moduli which can terminate on a non-Abelian monopole because of the matching of the moduli $\mathbb{C}P^{N-1}$ [48, 49], non-Abelian sine-Gordon kink here can terminate on a non-Abelian global vortex [50–53], see Ref. [4] as a review. We then promote the non-Abelian sine-Gordon solitons to those in non-Abelian $U(N)$ gauge theories with two N by N complex scalar fields coupled to each other by a non-Abelian extension of linear or quadratic Josephson interaction. The Abelian case reduces to phase solitons in two-gap superconductors [8–10], while the non-Abelian extension is relevant to a color superconductor of the color-flavor locking phase of dense QCD matter [4, 54].

This paper is organized as follows. In Sec. II, after reviewing sine-Gordon kinks in the conventional sine-Gordon model, we discuss non-Abelian sine-Gordon kinks in the $U(N)$ chiral Lagrangian. In Sec. III, sine-Gordon kinks with a modified mass term and their non-Abelian $U(N)$ generalization are discussed. In Sec. IV, these sine-Gordon kinks are promoted to gauge theories. The $U(1)$ gauge theory is nothing but two-gap superconductors or chiral p-wave superconductors corresponding to the conventional or modified mass term, respectively. In Sec. V, we discuss that a sine-Gordon kink can terminate on a non-Abelian global vortex. Sec. VI is devoted to summary and discussion.

II. THE SINE-GORDON MODEL AND CHIRAL LAGRANGIAN

A. The sine-Gordon model

The sine-Gordon kink is characterized by the first homotopy group $\pi_1[U(1)] \simeq \mathbb{Z}$. The Lagrangian density of conventional sine-Gordon model is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\theta)^2 - m^2(1 - \cos\theta) \quad (1)$$

with $\mu = 0, 1, \dots, d-1$ and $0 \leq \theta < 2\pi$. We consider static configurations depending on one spatial direction x . The static energy density is

$$\mathcal{E} = \frac{1}{2}(\partial_x\theta)^2 + m^2(1 - \cos\theta). \quad (2)$$

The Bogomol'nyi completion for the energy density is obtained as

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}(\partial_x \theta)^2 + 2m^2 \sin^2 \frac{\theta}{2} \\
&= \frac{1}{2} \left(\partial_x \theta \mp 2m \sin \frac{\theta}{2} \right)^2 \pm 2m \partial_x \theta \sin \frac{\theta}{2} \\
&\geq \left| 2m \partial_x \theta \sin \frac{\theta}{2} \right| = |t_{\text{SG}}|
\end{aligned} \tag{3}$$

with the topological charge density defined by

$$t_{\text{SG}} \equiv 2m \partial_x \theta \sin \frac{\theta}{2} = -4m \partial_x \left(\cos \frac{\theta}{2} \right). \tag{4}$$

The inequality is saturated by the BPS equation

$$\partial_x \theta \mp 2m \sin \frac{\theta}{2} = 0. \tag{5}$$

A single-kink solution interpolating between $\theta = 0$ at $x \rightarrow -\infty$ to $\theta = 2\pi$ at $x \rightarrow +\infty$ can be given as

$$\theta(x) = 4 \arctan \exp m(x - X) \tag{6}$$

with the position X in the x -coordinate. The topological charge for this solution is

$$T_{\text{SG}} = \int dx t_{\text{SG}} = -4m \left[\cos \frac{\theta}{2} \right]_{x=-\infty}^{x=+\infty} = -4m(-1 - 1) = 8m. \tag{7}$$

The width of the sine-Gordon kink is $1/m$.

For later convenience, we introduce a new variable taking a value in the $U(1)$ group by

$$u \equiv e^{i\theta}. \tag{8}$$

From $\partial_x \theta = -(i/2)(u^* \partial_x u - (\partial_x u^*)u)$, the BPS equation is rewritten as

$$-\frac{i}{2}(u^* \partial_x u - (\partial_x u^*)u) \mp m \sqrt{2 - u - u^*} = 0 \tag{9}$$

and the topological charge density is rewritten as

$$t_{U(1)} = -\frac{im}{2}(u^* \partial_x u - (\partial_x u^*)u) \sqrt{2 - u - u^*} = -2m \partial_x \left(\sqrt{2 + u + u^*} \right) \tag{10}$$

The single-kink solution is

$$u(x) = \exp(4i \arctan \exp[m(x - X)]) \tag{11}$$

with the boundary condition $u \rightarrow 1$ for $x \rightarrow \pm\infty$.

B. Non-Abelian sine-Gordon model as chiral Lagrangian

Here we consider the $U(N)$ group:

$$U(x) \in U(N) \simeq \frac{U(1) \times SU(N)}{\mathbb{Z}_N} \quad (12)$$

with the first homotopy group is nontrivial:

$$\pi_1[U(N)] = \mathbb{Z}. \quad (13)$$

The Lagrangian for a $U(N)$ principal chiral model (chiral Lagrangian) for a $U(N)$ -valued field $U(x)$ is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \text{tr} \partial_\mu U^\dagger \partial^\mu U - \frac{m^2}{2} \text{tr} (2\mathbf{1}_N - U - U^\dagger) \\ &= \frac{1}{2} \text{tr} (iU^\dagger \partial_\mu U)^2 - \frac{m^2}{2} \text{tr} (2\mathbf{1}_N - U - U^\dagger). \end{aligned} \quad (14)$$

This Lagrangian is invariant under the chiral $SU(N)_L \times SU(N)_R$ symmetry

$$U(x) \rightarrow V_L U(x) V_R^\dagger, \quad V_{L,R} \in SU(N)_{L,R} \quad (15)$$

The Lagrangian admits the unique vacuum $U = \mathbf{1}_N$. The chiral symmetry is spontaneously broken to the vector-like symmetry

$$U(x) \rightarrow V U(x) V^\dagger, \quad V \in SU(N)_{L+R=V}. \quad (16)$$

The energy density for static configuration and its Bogomol'nyi completion are given as

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \text{tr} (iU^\dagger \partial_x U)^2 - \frac{m^2}{2} \text{tr} (2\mathbf{1}_N - U - U^\dagger) \\ &= \frac{1}{2} \text{tr} \left[-\frac{i}{2} (U^\dagger \partial_x U - \partial_x U^\dagger U) \mp m \sqrt{2\mathbf{1}_N - U - U^\dagger} \right]^2 \\ &\quad \pm \frac{m}{2} \text{tr} \left[-\frac{i}{2} (U^\dagger \partial_x U - \partial_x U^\dagger U) \sqrt{2\mathbf{1}_N - U - U^\dagger} \right] \\ &\geq |t_{U(N)}|, \end{aligned} \quad (17)$$

with the topological charge, defined by

$$t_{U(N)} \equiv -\frac{m}{2} \text{tr} \left[i(U^\dagger \partial_x U - \partial_x U^\dagger U) \sqrt{2\mathbf{1}_N - U - U^\dagger} \right]. \quad (18)$$

The BPS equation is obtained as

$$-\frac{i}{2} (U^\dagger \partial_x U - \partial_x U^\dagger U) \mp m \sqrt{2\mathbf{1}_N - U - U^\dagger} = \mathbf{0}_N. \quad (19)$$

This equation is invariant under the $SU(N)$ symmetry in Eq. (16).

Let us construct solutions to this equation. The simplest ansatz is given by the following *Abelian* solution

$$U(x) = u(x)\mathbf{1}_N. \quad (20)$$

By substituting this ansatz into Eq. (19), we find that $u(x)$ again satisfies Eq. (9). The tension (energy per unit area) of this configuration is $T = NT_{\text{SG}}$.

Next, we construct *non-Abelian* solutions. Let us consider the following ansatz [37]:

$$U(x) = \text{diag}(u(x), 1, \dots, 1) \quad (21)$$

By substituting this ansatz into Eq. (19), we find that $u(x)$ satisfies Eq. (9) and the one-kink solution is obtained as Eq. (11). The tension of this configuration is $T = T_{\text{SG}}$. Although the solution is obtained by embedding the Abelian solution into the upper-left corner, this solution is truly non-Abelian; In terms of group elements, the ansatz in Eq. (21) can be rewritten as

$$\begin{aligned} U(x) &= \exp\left(i\frac{\theta(x)}{N}\right) \exp(i\theta(x)T_0), \\ T_0 &\equiv \frac{1}{N}\text{diag.}(N-1, -1, \dots, -1). \end{aligned} \quad (22)$$

From this expression, one can see that the $U(1)$ group element rotates only $2\pi/N$ while the rest is compensated by an $SU(N)$ group element T_0 . Namely at $x = \infty$ ($\theta = 2\pi$) the $U(1)$ group element becomes $\exp(i\frac{2\pi}{N}) = \omega$ while the $SU(N)$ group element becomes $\exp(2\pi iT_0) = \text{diag.}(\omega^{N-1}, \omega^{-1}, \dots, \omega^{-1}) = \omega^{-1}\mathbf{1}_N$. The $SU(N)$ group element connects the trivial element to an element of the center \mathbb{Z}_N of the $SU(N)$ group.

There is a continuous degeneracy of the solutions with the same energy. Since the Lagrangian and the BPS equation is invariant under the $SU(N)$ transformation in Eq. (16), the most general solution is obtained as

$$U(x) = V\text{diag}(u(x), 1, \dots, 1)V^\dagger, \quad V \in SU(N). \quad (23)$$

Since there exists a redundancy for the action of V , V in fact takes a value in the coset space

$$V \in \frac{SU(N)}{SU(N-1) \times U(1)} \simeq \mathbb{C}P^{N-1}. \quad (24)$$

Therefore, the one-kink solution has the moduli

$$\mathcal{M} = \mathbb{R} \times \mathbb{C}P^{N-1}. \quad (25)$$

In terms of the group elements, the general solution can be rewritten as

$$\begin{aligned} U(x) &= \exp\left(i\frac{\theta(x)}{N}\right) \exp(i\theta(x)VT_0V^\dagger) \\ &= \exp\left(i\frac{\theta(x)}{N}\right) \exp i\frac{\theta(x)}{N}T, \end{aligned} \quad (26)$$

with $T \equiv VT_0V^\dagger$. T can be any $SU(N)$ generator normalized as $e^{i2\pi T} = \omega^{-1}\mathbf{1}_N$.

Let us introduce the orientational vector $\phi \in \mathbb{C}^N$ with a constraint

$$\phi^\dagger\phi = 1, \quad (27)$$

which represents homogeneous coordinates of $\mathbb{C}P^{N-1}$. The generator T and the general solution in Eq. (26) can be rewritten by using the orientational vector as

$$T = VT_0V^\dagger = \phi\phi^\dagger - \frac{1}{N}\mathbf{1}_N, \quad (28)$$

$$U(x) = \exp(i\theta(x)\phi\phi^\dagger). \quad (29)$$

III. THE MODIFIED SINE-GORDON MODEL AND CHIRAL LAGRANGIAN

A. The modified sine-Gordon model

We consider the Lagrangian density of a sine-Gordon model with an unconventional potential, given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\theta)^2 - m^2(1 - \cos^2\theta) \quad (30)$$

with $\mu = 0, 1$. This model admits two vacua $\theta = 0, \pi$ in the defined range $0 \leq \theta \leq 2\pi$. We concentrate on static configurations. The static energy density is

$$\mathcal{E} = \frac{1}{2}(\partial_x\theta)^2 + m^2(1 - \cos^2\theta) = \frac{1}{2}(\partial_x\theta)^2 + m^2\sin^2\theta. \quad (31)$$

The Bogomol'nyi completion for the energy density is obtained as

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}[(\partial_x\theta)^2 + 2m^2\sin^2\theta] \\ &= \frac{1}{2}\left(\partial_x\theta \mp \sqrt{2}m\sin\theta\right)^2 \pm \sqrt{2}m\partial_x\theta\sin\theta \\ &\geq \left|\sqrt{2}m\partial_x\theta\sin\theta\right| = |t_{\text{SG}}| \end{aligned} \quad (32)$$

with the topological charge density

$$t_{\text{SG}} \equiv \sqrt{2}m\partial_x\theta \sin\theta = -\sqrt{2}m\partial_x(\cos\theta). \quad (33)$$

The inequality is saturated by the BPS equation

$$\partial_x\theta \mp \sqrt{2}m \sin\theta = 0. \quad (34)$$

A one-kink solution interpolating between $\theta = 0$ at $x \rightarrow -\infty$ to $\theta = \pi$ at $x \rightarrow +\infty$ can be given as

$$\theta(x) = 2 \arctan \exp \sqrt{2}m(x - X) \quad (35)$$

with the position X in the x -coordinate and the width $1/m$. The topological charge for this solution is

$$T_{\text{SG}} = \int dx t_{\text{SG}} = -\sqrt{2}m [\cos\theta]_{x=-\infty}^{x=+\infty} = -\sqrt{2}m(-1 - 1) = 2\sqrt{2}m. \quad (36)$$

In terms of $u(x) = e^{i\theta(x)}$, the BPS equation is rewritten as

$$\begin{aligned} \partial_x u \mp \frac{\sqrt{2}}{2}m(1 - u^2) &= 0, \\ (\leftrightarrow -i(u^*\partial_x u - (\partial_x u^*)u) \mp \sqrt{2}im(u - u^*) &= 0), \end{aligned} \quad (37)$$

and the topological charge density is rewritten as

$$t_{\text{U(1)}} = -\frac{\sqrt{2}}{2}mu^*\partial_x u(u - u^*) \quad (38)$$

The one-kink solution is

$$u(x) = \exp \left(2i \arctan \exp \frac{\sqrt{2}m}{4}(x - X) \right). \quad (39)$$

B. Non-Abelian sine-Gordon model as chiral Lagrangian with modified mass

The Lagrangian for $U(N)$ principal chiral model with a modified mass is

$$\mathcal{L} = \frac{1}{2}\text{tr} \partial_\mu U^\dagger \partial^\mu U - V = \frac{1}{2}\text{tr} (iU^\dagger \partial_\mu U)^2 - V \quad (40)$$

$$\begin{aligned} V &= m^2 \text{tr} (2\mathbf{1}_N - U^2 - U^{\dagger 2}) = m^2 \text{tr} (2\mathbf{1}_N - U - U^\dagger)(2\mathbf{1}_N + U + U^\dagger) \\ &= m^2 \text{tr} (\mathbf{1}_N - U^2)(\mathbf{1}_N - U^{\dagger 2}). \end{aligned} \quad (41)$$

This model admits two vacua $U = \pm \mathbf{1}_N$. The energy density for static configuration and its Bogomol'nyi completion are given as

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2} \text{tr} \partial_x U^\dagger \partial_x U + m^2 \text{tr} (\mathbf{1}_N - U^2)(\mathbf{1}_N - U^{\dagger 2}) \\
&= \frac{1}{2} \text{tr} \left[\{ \partial_x U^\dagger \mp \sqrt{2}m(\mathbf{1}_N - U^{\dagger 2}) \} \{ \partial_x U \mp \sqrt{2}m(\mathbf{1}_N - U^2) \} \right] \\
&\quad \pm 2m \text{tr} [\partial_x U^\dagger (\mathbf{1}_N - U^2) + \partial_x U (\mathbf{1}_N - U^{\dagger 2})] \\
&\geq |t_{U(N)}|,
\end{aligned} \tag{42}$$

with the topological charge, defined by

$$\begin{aligned}
t_{U(N)} &\equiv \sqrt{2}m \text{tr} [\partial_x U^\dagger (\mathbf{1}_N - U^2) + \partial_x U (\mathbf{1}_N - U^{\dagger 2})] \\
&= \sqrt{2}m \text{tr} [U^\dagger \partial_x U (U - U^\dagger) + \text{h.c.}].
\end{aligned} \tag{43}$$

The BPS equation is obtained as

$$\begin{aligned}
&\partial_x U \mp \sqrt{2}m(\mathbf{1}_N - U^2) = \mathbf{0}_N \\
&\Leftrightarrow (iU^\dagger \partial_x U \mp \sqrt{2}im(U^\dagger - U)) = \mathbf{0}_N.
\end{aligned} \tag{44}$$

As in the same manner, the Abelian kink in Eq. (39) can be embedded into a conner as in Eq. (21) to obtain a non-Abelian kink. Also, it allows the $\mathbb{C}P^{N-1}$ moduli as Eq. (23).

IV. NON-ABELIAN SINE-GORDON SOLITON IN GAUGE THEORIES

A. Abelian gauge theory: two-gap superconductors and chiral p-wave superconductors

Let us consider a $U(1)$ gauge theory coupled with two complex scalar fields $\phi_i(x)$ ($i = 1, 2$), given by

$$\mathcal{L} = \frac{1}{2} \sum_{i=1,2} D_\mu \phi_i^* D^\mu \phi_i + \mathcal{L}_J - \sum_{i=1,2} \frac{\lambda_i}{4} (|\phi_i|^2 - 1)^2 + \frac{1}{4e^2} F_{\mu\nu}^2 \tag{45}$$

with $D_\mu \phi_i = (\partial_\mu - iA_\mu)\phi_i$. \mathcal{L}_J is a Josephson term either linear or quadratic:

$$\begin{aligned}
\mathcal{L}_{J,1} &= \frac{\gamma}{2} (\phi_1^* \phi_2 + \text{c.c.} - 2) \\
\mathcal{L}_{J,2} &= \frac{\gamma}{2} [(\phi_1^* \phi_2)^2 + \text{c.c.} - 2].
\end{aligned} \tag{46}$$

The gauge transformation is defined by

$$\phi_i \rightarrow e^{i\alpha(x)}\phi_i, \quad A_\mu \rightarrow A_\mu + \partial_\mu\alpha(x), \quad (47)$$

while a $U(1)$ global transformation

$$\phi_1 \rightarrow e^{i\beta}\phi_1, \quad \phi_2 \rightarrow e^{-i\beta}\phi_2 \quad (48)$$

is explicitly broken by $\gamma \neq 0$.

Let us take strong coupling limit (with keeping γ finite):

$$e, \lambda_i \rightarrow \infty, \quad (49)$$

giving constraints

$$|\phi_i| = 1, \quad \phi_i = e^{i\theta_i}. \quad (50)$$

With taking a gauge $A_\mu = \partial_\mu\theta_2$ and defining the phase difference $\theta(x) \equiv \theta_1(x) - \theta_2(x)$, the covariant derivative terms in Lagrangian in Eq. (45) become

$$\begin{aligned} D_\mu\phi_1 &= i(\partial_\mu\theta_1 - A_\mu)e^{i\theta_1} = i\partial_\mu(\theta_1 - \theta_2)e^{i\theta_1} = i\partial_\mu\theta e^{i\theta_1}, \\ D_\mu\phi_2 &= i(\partial_\mu\theta_2 - A_\mu)e^{i\theta_2} = 0, \end{aligned} \quad (51)$$

while the Josephson terms in Eq. (46) become

$$\mathcal{L}_{J,1} = -m^2(1 - \cos\theta), \quad \mathcal{L}_{J,2} = -m^2(1 - \cos^2\theta), \quad \gamma \equiv m^2. \quad (52)$$

The gauge theory Lagrangian in Eq. (45) reduces the sine-Gordon model in Eq. (1) or the modified sine-Gordon model in Eq. (30).

Let us remark on physical realizations of this model and its sine-Gordon solitons. A non-relativistic version of the Lagrangian has the kinetic and gradient terms

$$\frac{1}{2} \sum_i (i\phi_i^* D_0\phi_i + \text{h.c} - D_a\phi_i^* D_a\phi_i). \quad (53)$$

instead of the first term in the Lagrangian in Eq. (45). Here $a = 1, 2, (3)$ is a spatial index. The linear Josephson term $\mathcal{L}_{J,1}$ in Eq. (46) is relevant for the Landau-Ginzburg description of two-gap superconductors such as MgB_2 , in which the term proportional to γ is called the (internal) Josephson coupling and $\theta(x)$ is called the Leggett mode. The sine-Gordon

soliton is called the phase soliton in this context, which was first pointed out theoretically [8] and was found experimentally [9]. It is also relevant for a Josephson junction of two superconductors. On the other hand, the case with the quadratic Josephson interaction $\mathcal{L}_{J,2}$ in Eq. (46) is relevant for chiral p-wave superconductors [11], such as Sr_2RuO_4 .

A non-relativistic version of the Lagrangian (53) in which overall $U(1)$ is not gauged ($e = 0$) yields the Gross-Pitaevskii equation for two-component Bose-Einstein condensates of ultracold atomic gases such as Rb_{87} , in which the term proportional to γ is called a Rabi oscillation term. (In addition, the term $g_{12}|\phi_1|^2|\phi_2|^2$ is also present but it is not important for the phase solitons.) The sine-Gordon (phase) soliton in this case was studied in Ref. [12].

B. Non-Abelian gauge theory

Let us consider a $U(N)$ gauge theory coupled with two $N \times N$ matrix-valued complex scalar fields $\Phi_i(x)$ ($i = 1, 2$), whose Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \sum_{i=1,2} \text{tr} D_\mu \Phi_i^\dagger D^\mu \Phi_i + \frac{\gamma}{2} \text{tr} (\Phi_1^\dagger \Phi_2 + \text{h.c.} - 2\mathbf{1}_N) \\ & - \sum_{i=1,2} \frac{\lambda_i}{4} \text{tr} (\Phi_i^\dagger \Phi_i - \mathbf{1}_N)^2 + \frac{1}{4g^2} \text{tr} F_{\mu\nu}^2 \end{aligned} \quad (54)$$

with $D_\mu \Phi_i = (\partial_\mu - iA_\mu) \Phi_i$ and $A_\mu = A_\mu^A(x) T_A$ with $U(N)$ generators T_A . \mathcal{L}_J is a non-Abelian Josephson term either linear or quadratic:

$$\begin{aligned} \mathcal{L}_{J,1} &= \frac{\gamma}{2} \text{tr} (\Phi_1^\dagger \Phi_2 + \text{h.c.} - 2\mathbf{1}_N), \\ \mathcal{L}_{J,2} &= \frac{\gamma}{2} \text{tr} [(\Phi_1^\dagger \Phi_2)^2 + \text{h.c.} - 2\mathbf{1}_N]. \end{aligned} \quad (55)$$

The $U(N)_V$ gauge transformation is defined by

$$\Phi_i \rightarrow V(x) \Phi_i, \quad A_\mu \rightarrow V(x) A_\mu V(x)^{-1} + iV(x) \partial_\mu V^{-1}(x), \quad (56)$$

while a $U(N)_A$ global transformation

$$\Phi_1 \rightarrow g \Phi_1, \quad \Phi_2 \rightarrow g^{-1} \Phi_2, \quad g \in U(N)_A \quad (57)$$

is explicitly broken by $\gamma \neq 0$.

Let us take strong coupling limit (with keeping γ finite):

$$g, \lambda_i \rightarrow \infty, \quad (58)$$

giving constraints

$$\Phi_i^\dagger \Phi_i = \mathbf{1}_N. \quad (59)$$

These constraints can be solved as

$$\Phi_1(x) = \hat{U}(x), \quad \Phi_2(x) = \hat{U}^\dagger(x), \quad \hat{U}(x) \in U(N). \quad (60)$$

With taking a gauge $A_\mu = i\hat{U}^\dagger \partial_\mu \hat{U}$ and defining $U(x) \equiv \hat{U}^2(x)$, the covariant derivative terms in Lagrangian in Eq. (54) become

$$D_\mu \Phi_1 = \partial_\mu \hat{U} - iA_\mu \hat{U} = \partial_\mu U(x), \quad D_\mu \Phi_2 = \partial_\mu \hat{U}^\dagger - iA_\mu \hat{U}^\dagger = 0, \quad (61)$$

and the Josephson terms reduce to

$$\begin{aligned} \mathcal{L}_{J,1} &= -m^2 \text{tr} (2\mathbf{1}_N - U - U^\dagger), \\ \mathcal{L}_{J,2} &= -m^2 \text{tr} (2\mathbf{1}_N - U^2 - U^{\dagger 2}), \\ \gamma &\equiv m^2. \end{aligned} \quad (62)$$

Therefore, the gauge theory Lagrangian in Eq. (54) reduces the non-Abelian sine-Gordon model in Eq. (14) or the modified non-Abelian sine-Gordon model in Eq. (30).

The relativistic Lagrangian in Eq. (54) is relevant for a linear model description of chiral Lagrangian using a hidden local gauge symmetry for which gauge bosons of $U(N)$ gauge symmetry is vector mesons of the hidden local symmetry, see, e. g. Ref. [55].

A non-relativistic version of the Lagrangian has the kinetic and gradient terms

$$\frac{1}{2} \sum_i \text{tr} (i\Phi_i^\dagger D_0 \Phi_i + \text{h.c} - D_a \Phi_i^* D_a \Phi_i) \quad (63)$$

instead of the first term in the Lagrangian in Eq. (54). The non-relativistic case with $N = 3$ with ungauged $U(1)$ is relevant for the Landau-Ginzburg description of the color-flavor locking phase (a color superconductor) for high density QCD [4, 54]. In this case, $(\Phi_1)_{\alpha i} = \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} q_{j\beta}^L q_{k\gamma}^L$ and $(\Phi_2)_{\alpha i} = \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} q_{j\beta}^R q_{k\gamma}^R$ are diquark condensates of left and right handed quarks $q_{j\beta}^L$ and $q_{j\beta}^R$, respectively, where $\alpha, \beta, \gamma = 1, 2, 3$ and $i, j, k = 1, 2, 3$ are color and flavor indices, respectively.

Here, we have considered the potential for the $U(1)$ symmetry induced from quark mass in chiral Lagrangian in QCD. On the other hand, there is another potential term $V \sim$

$\det \Phi_1 + \det \Phi_2$ induced from the $U(1)_A$ anomaly at quantum level. The non-Abelian sine-Gordon kink should be deformed by this potential accordingly [56]. Therefore, in real QCD, our solutions are relevant in asymptotically high temperature or high density, in which the $U(1)_A$ anomaly disappears.

V. NON-ABELIAN VORTEX THAT TERMINATES NON-ABELIAN SINE-GORDON KINK

The $U(N)$ chiral Lagrangian or more precisely the corresponding $U(N)$ linear sigma model admits a non-Abelian global vortex [50–53], see Ref. [4] as a review. When one discusses the asymptotic form of the vortex solution, the chiral Lagrangian is enough. Here, we briefly discuss a relation between the non-Abelian global vortex and the non-Abelian sine-Gordon kink.

Let (r, φ, z) be cylindrical coordinates of space. Then, the asymptotic form of a non-Abelian global vortex can be written as

$$U(r \rightarrow \infty, \varphi, z) = \text{diag}(e^{i\theta(\varphi)}, 1, \dots, 1). \quad (64)$$

In the limit of no mass term ($m = 0$), the unit winding solution is simply given by $\theta = \varphi$ so that the vortex is axisymmetric. The configuration in Eq. (64) can be rewritten as

$$\begin{aligned} U(r \rightarrow \infty, \varphi, z) &= \exp\left(i\frac{\theta(\varphi)}{N}\right) \exp(i\theta(x)T_0), \\ T_0 &\equiv \frac{1}{N}\text{diag.}(N-1, -1, \dots, -1). \end{aligned} \quad (65)$$

It is obvious that the configuration of the vortex breaks the $SU(N)_V$ symmetry of the vacuum to a subgroup $SU(N-1) \times U(1)$ so that there appear moduli $\mathbb{C}P^{N-1}$, although these moduli are non-normalizable [51, 52].

In the presence of the mass term ($m \neq 0$), the global vortex configuration is deformed and is no more axisymmetric. In this case, the potential term appears for the field $\theta(\varphi)$ in the vortex ansatz in Eq. (64). This is of course the sine-Gordon potential discussed in the previous sections. Only the difference is the argument of θ is $\theta(\varphi)$ here and $\theta(x)$ before. The final configuration is a non-Abelian vortex attached by a non-Abelian sine-Gordon kink, as schematically drawn in Fig. 1. Both the non-Abelian vortex and non-Abelian sine-Gordon kink have the $\mathbb{C}P^{N-1}$ moduli, and consequently they match at a junction line [65].

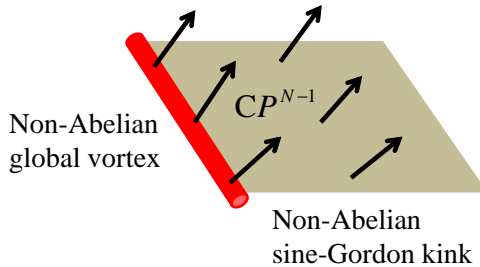


FIG. 1: A junction of a non-Abelian vortex and a non-Abelian sine-Gordon domain wall. A non-Abelian vortex is attached by a non-Abelian sine-Gordon kink in the presence of the mass. In other words, the latter can terminate on the former. The $\mathbb{C}P^{N-1}$ moduli, that are denoted by arrows, match at the junction line.

This fact implies the instability of sine-Gordon kinks in the $U(N)$ *linear* sigma model as in the same manner with an axion string [5]. In $d = 2 + 1$, the sine-Gordon domain line can terminate on a global non-Abelian vortex [4, 50–53]. The domain line can decay by creating a pair of a non-Abelian vortex and a non-Abelian anti-vortex, as shown in Fig. 2 (a). In $d = 3 + 1$, the non-Abelian sine-Gordon domain wall can decay by creating a hole bound by a closed non-Abelian vortex string, as illustrated in Fig. 2 (b). This process can occur either thermally or by quantum tunneling. More details will be discussed elsewhere. However, note that the instability does not exist in the nonlinear model, the $U(N)$ chiral Lagrangian. This is the same situation with an axion string [5].

VI. SUMMARY AND DISCUSSION

We have pointed out that the $U(N)$ chiral Lagrangian admits a non-Abelian sine-Gordon kink that carries non-Abelian moduli $\mathbb{C}P^{N-1} \simeq SU(N)/[SU(N-1) \times U(1)]$. We have also presented the non-Abelian gauge theory that admits the same non-Abelian sine-Gordon kink. In the Abelian case, this reduces to the Lagrangian for two-gap superconductors. Two possibilities to realize it in QCD have been discussed. We have also briefly discussed in the $U(N)$ linear sigma model that a sine-Gordon kink can terminate on a non-Abelian global vortex, implying the instability of the sine-Gordon kink in the linear model.

Several discussions are addressed here. One of the most important task remaining is

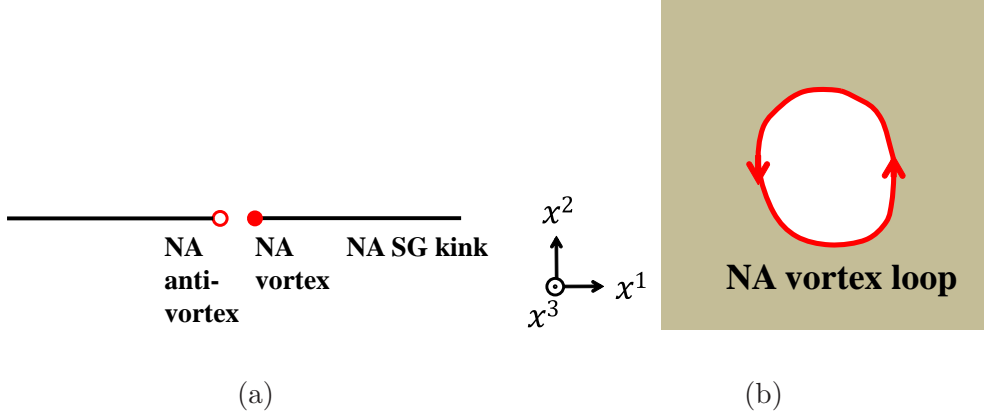


FIG. 2: Decay of a non-Abelian sine-Gordon kink. (a) In $d = 2 + 1$, a non-Abelian sine-Gordon domain line can decay by creating a pair of a non-Abelian vortex and a non-Abelian anti-vortex. (b) In $d = 3 + 1$ the non-Abelian sine-Gordon domain wall can decay by creating a hole bound by a closed non-Abelian vortex string. These processes can occur either thermally or by quantum tunneling.

constructing the low-energy effective theory by the moduli approximation [57], which is the $\mathbb{C}P^{N-1}$ model. One then can construct $\mathbb{C}P^{N-1}$ lumps on it that would represent $U(N)$ Skyrmions as was so for the $SU(2)$ model with two vacua [27, 28]. See Ref. [58] for a further study along this line.

The interaction between two kinks located at $x = X_{1,2}$ with the orientations $\phi_{1,2}$ can be considered. Like the Abrikosov-type ansatz for vortices, we can give an ansatz for the total configuration as $U_{\text{tot}}(x) = U_1(x - X_1, \phi_1)U_2(x - X_2, \phi_2)$ for well-separated kinks $|X_1 - X_2| \gg m^{-1}$. In particular, an Abelian sine-Gordon kink would be separated into N non-Abelian kinks without cost of energy, which can be expected from the fact that an Abelian kink has energy N multiple of those of non-Abelian kinks. A similar calculation was done for the force between two non-Abelian global vortices [4, 52].

In two-gap superconductors, a unit winding vortex can be split into two fractional vortices winding around different components, which are connected by a sine-Gordon kink [10, 17, 18]. The same happens for coherently coupled multi-component BECs [19–21]. In the same way, a local non-Abelian vortex can be split into a set of two global non-Abelian vortices connected by a non-Abelian sine-Gordon domain wall discussed here. In the case of the color-flavor locked phase of dense quark matter, a non-Abelian vortex [4, 59] has $1/3$ fractional $U(1)$

winding in both Φ_1 and Φ_2 , but it may be decomposed into a global vortex with $1/6$ $U(1)$ winding ($1/3$ $U(1)$ winding in only one of Φ_1 and Φ_2). This will be also discussed elsewhere.

The $U(N)$ principal chiral model studied in this paper has been found to appear as the effective theory of a non-Abelian domain wall [60]. If a Josephson term is added in the bulk theory, this domain wall behaves as a Josephson junction of two color superconductors and the mass term is induced in the $U(N)$ principal chiral model on the wall [61]. Then, non-Abelian sine-Gordon solitons describe non-Abelian Josephson vortices, that is, non-Abelian vortices trapped inside the Josephson junction [61, 62].

Non-Abelian $U(N)$ Sine-Gordon kinks can be extended to the case of arbitrary gauge groups G in the form of $\frac{G \times U(1)}{\mathbb{Z}_r}$ with the center \mathbb{Z}_r of G , since non-Abelian vortices with this type of gauge groups were studied before [63], such as $SO(N)$ and $USp(2N)$ groups [64].

Finally, the sine-Gordon model is integrable. Therefore, we expect the non-Abelian sine-Gordon model presented here is also integrable.

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